A TERM PAPER ON MONTE CARLO ANALYSIS / SIMULATION

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1.0 INTRODUCTION

Monte Carlo (MC) approach to analysis was developed in the 1940's, it is a computer based analytical method which employs statistical sampling techniques for obtaining a probabilistic approximation to the solution of a mathematical equation or model by utilizing sequences of random numbers as inputs into a model which yields results that are indications of the performance of the developed model.

Monte Carlo simulation was developed as part of the atomic program. Scientist at the Los Alamos National Laboratory originally used it to model the random diffusion of neutrons. The scientist who developed this simulation technique gave it the name 'Monte Carlo" after the city in Monaco and its many casinos. Monte Carlo simulation are used in a wide array of applications, including physics, finance, and system reliability

Monte Carlo analysis utilizes statistical tools to mathematically model a real life system or process and then it estimates the probability of obtaining a successful outcome. The statistical distribution of the process to be modeled must be determined first before Monte Carlo simulation can be applied.

Monte Carlo entails using random numbers as a tool to compute something that is not random. MC simulation is a versatile tool to analyze and evaluate complex measurements using a model of a system, and to experiment with the model towards drawing inferences of the system's behavior.

QUESTIONS TO ASK WHEN CONSIDERING USING MONTE CARLO

1. Consider the problem. Does it have many sources of uncertainty?
2. Is there an analytical solution? If so, use that.
3. Choose the software you will use for Monte Carlo simulation.
4. Decide on the nature and source of inputs to use
5. What distribution do the random variables have?
6. How do we generate these random variables for the simulation?
7. How do we analyze the output of the simulation?
8. How many simulation runs do we need?
9. How do we improve the efficiency of the simulation?
10. How do we interpret the generated result?

ATTRIBUTES OF MONTE CARLO SIMULATION

There is no single Monte Carlo method—the term covers a wide range of approaches to simulation. However, these approaches use a certain pattern in which:

1. A domain of possible inputs is defined;
2. Inputs are randomly generated from the domain;
3. Using the inputs, a deterministic computation is performed;
4. The results are aggregated from the individual computations to give a final result

2.0 MONTE CARLO ANALYSIS

PROCEDURE FOR APPLYING MONTE CARLO

1. Determine the pseudo-population or model that represents the true population of interest.
2. Use a sampling procedure to sample from the pseudo-population.
3. Calculate a value for the statistic of interest and store it.
4. Repeat steps 2 and 3 for N trials.
5. Use the N values found in step 4 to study the distribution of the statistic.

It is often asked that when do we simulate? and when do we approach a problem using analytical methods? The catch is this '*Calculate when you can, simulate when you can't*!"

One favorable feature of Monte Carlo is that it is possible to estimate the order of magnitude of statistical error, which is the dominant error in most Monte Carlo computations. These estimates are often called *error bars* because of the way they are indicated on plots of Monte Carlo results. Monte Carlo error bars are essentially statistical confidence intervals. At times algorithm used for MC simulations could result in wide variations in output for different computations and this necessitates developing better algorithms. The search for more accurate alternative algorithms is often called *variance reduction*.

ADVANTAGES

1. Using Monte Carlo simulation is quite straightforward.
2. It can provide statistical sampling for numerical experiments using a computer.
3. In optimization problems, Monte Carlo simulation can often reach the optimum and overcome local extremes.
4. It provides approximate solutions to many mathematical problems.
5. Monte Carlo analysis produces a narrower range of results than a “what if” analysis.

DISADVANTAGES

1. Monte Carlo simulation is not universally accepted in simulating a system that is not in equilibrium (i.e. in a transient state).
2. A large number of samples are required to reach the desired results. This can be time-consuming compared to using a spreadsheet program, such as Excel, which can generate a simple calculation fairly quickly.
3. A single sample cannot be used in simulation; to obtain results there must be many samples.
4. The results are only an approximation of the true value.
5. Simulation results can show large variance.
6. It may be very expensive to build simulation
7. It is easy to misuse simulation by stretching it beyond the limits of credibility

COMPARISON OF ANALYTICAL AND MONTE CARLO SIMULATION MODELS

Simulation Method		
Advantages	**Analytical**	**Monte-Carlo**
	a. Gives exact results (given the assumptions of the model).	a. Very flexible. There is virtually no limit to the analysis. Empirical distributions can be handled.
	b. Once the model is developed, output will generally be rapidly obtained.	b. Can generally be easily extended and developed as required.
	c. It need not always be implemented on a computer – paper analyses may suffice.	c. Easily understood by non-mathematicians.
Disadvantages	a. Generally requires restrictive assumptions to make the problem tractable.	a. Usually requires a computer.
	b. Because of a. it is less flexible than Monte-Carlo. In particular, the scope for extending or developing a model may be limited.	b. Calculations can take much longer than analytical models.
	c. The model might only be understood by mathematicians. This may cause credibility problems if output conflicts with preconceived ideas of designers or management.	c. Solutions are not exact, but depend on the number of repeated runs used to produce the output statistics. That is, all outputs are estimates.

COMMON AREAS OF APPLICATION OF MONTE CARLO

1. Monte Carlo Simulation is used in Mathematics and Statistical Physics
   It is used for numerically solving complex multi-dimensional partial differentiation and integration problems. It is also used to solve optimization problems in Operations Research (these optimization methods are called simulation optimization). MC method is used for simulating quantum systems, which allows a direct representation of many-body effects in the quantum domain, at the cost of statistical uncertainty that can be reduced with more simulation

time. One of the most famous early uses of MC simulation was by Enrico Fermi in 1930, when he used a random method to calculate the properties of the newly-discovered neutron

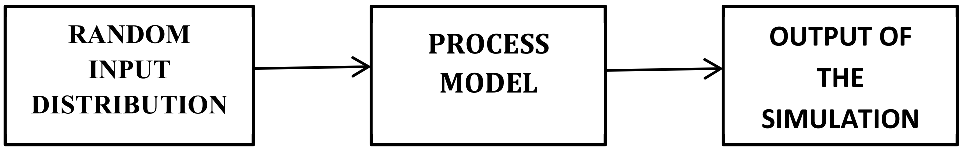
2. Monte Carlo Simulation in Engineering
   It is used in reliability analysis of components in engineering and to determine the effective life of pressure vessels in reactors. Also in electronics engineering and circuit design; circuits in computer chips are simulated using MC methods for estimating the probability of fetching instructions in memory buffers etc.

3. It is also used in finance in the following areas:
   Real Options Analysis, Portfolio Analysis, Option Analysis, Personal Financial Planning etc

4. Monte Carlo Simulation is also used in Reliability Analysis and Six Sigma

3.0 SAMPLE APPLICATION OF MONTE CARLO SIMULATION

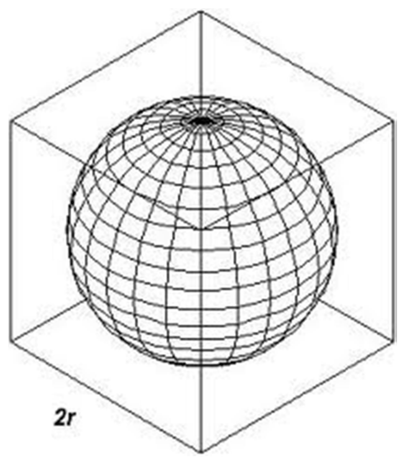
In basic terms, Monte Carlo entails taking a large set of defined random inputs, applying them as inputs to a model of the process under study and studying the result obtained as shown below.



Using the above steps lets determine the Volume of a unit sphere (radius =1)

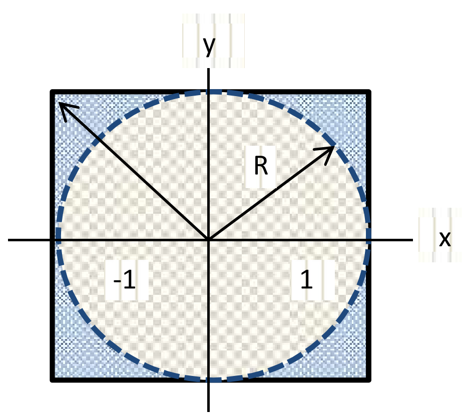
Finding the Volume of a unit sphere (Radius = R = 1)

To determine the volume, let us assume that the sphere is inscribed in a cube of length = 2R



In a 3D space, the distance (d) between any two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



The central plane of the sphere showing the x and y axes

Let the centre of the sphere be at (0, 0, 0) hence,

$$d^2 = x^2 + y^2 + z^2$$

Let us define the function f (x, y) = z $= \sqrt{(r)^2 - (x)^2 + (y)^2}$

Using Monte Carlo Integration (Hit and miss method)

For an M+1 dimensional volume; Integral $V^{M+1} = V^M * f$

Where V^M is the m-dimensional volume defining the integration area

f is the mean value $= \frac{1}{N}\sum_{i=1}^{N} f(x)_i = \frac{N_h}{N}$

N is the total number of points randomly considered or the number of trials.

N_h is the number of hits

NOTE: - The integral over f(x, y) only covers half the volume of the sphere so we multiply by 2

Actual Integral = 2* Integral Volume * f
Integral Volume = pi * r^2

The mean of f(x, y) is obtained by considering the portion of the total N points that falls within the circular region within the square such that;

f = sum of the values of f(x, y) within the circle ÷ the number of points within the circle

Using MATLAB

```
Clear
clc
n=10000;
x =rand (n,1);
y=rand(n,1);
withinsphere=0;  z= zeros(); d=zeros();
r=1; sum=0;

for i=1:n
    d(i)= x(i).^2 + y(i).^2;
    if d(i)<=r^2 && d(i)>0
        z(i)= sqrt (r^2- d(i));
        sum = sum + z(i);
        withinsphere= withinsphere + 1;
   end
```

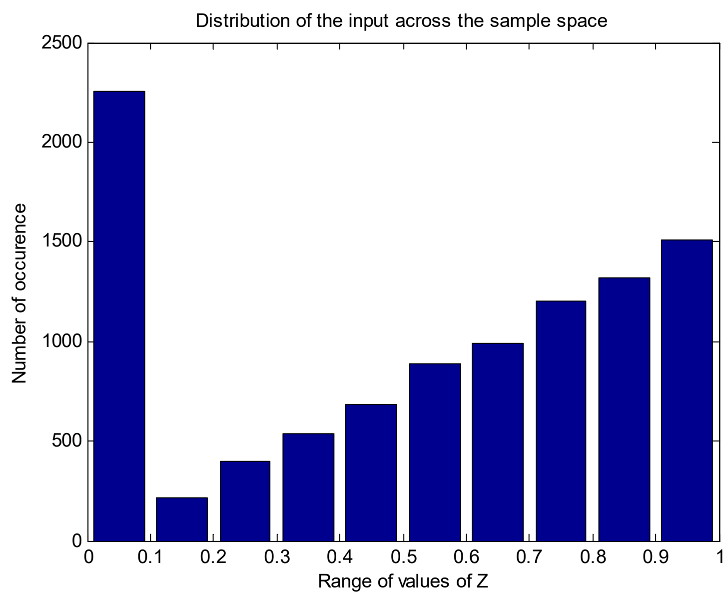
```
end

meanz= sum/withinsphere;
%since the integral over f(x,y) = z =sqrt [r2 - [2 + y2]] only covers
half the volume
%of the sphere we multiply by 2
%The volume by monte carlo integration = Actual Integral =
% 2* pi*r^2 * meanz
%where pi*r*r is now the integral volume
%expected answer is 4.18879

spherevolume =  2* pi*r*r * meanz;
RESULT = sprintf('Out of %d random points %d fell within the sphere
giving a simulated sphere volume of %d',n, withinsphere,
spherevolume)
disp('By calculation the expected values is 4.1889020')
```

Discussion

For N=100000 MATLAB gave a value of 4.187988e+000 which is very close to the expected value of $4\pi / 3 = 4.1889020$



Using a Sphere inscribed in a Cube to find the value of pi

Using this simulation we aim to confirm if there is any relationship between the accuracy of the simulation result and the number of random inputs used vis-à-vis the number of times for which the simulation was repeated.

Analytically

The volume of a sphere is given by 4/3 π r^3

For the cube (L = B = W = 2r); Volume = $(2r)^3$

The ratio of the two volumes gives

$$\frac{Sphere\ volume}{Cube\ volume} = \frac{\frac{4}{3}\pi r^3}{8r^3} = \frac{\pi}{6} = 0.523599$$

Using MATLAB

```
% here the ratio of sphere volume to cube is pi/6
%The distance from a point in a 3d space is r^2= sqrt [x^2 + y^2 +z^2]

factor=10000;
n=6 *factor;
x =rand (n,1);
y=rand(n,1);
z=rand(n,1);
k=0;  T= [];
% repeating the iteration T times and finding the average to improve
% accuracy
range=1;
for T=1:range
for i=1:n
    d(i)= sqrt (x(i).^2 + y(i).^2 + z(i).^2);
    if d(i)<=1 && d(i)>0
        k=k+1;
   end
end
store(T)=k/factor;
k=0;
end
 %to find the average value of pi
 sum=0;
 for T=1:range
     sum= sum + store (T);
 end
      pii = sum/range
```

Observation

After running the algorithm for different values of N for several numbers of times and then finding the mean and standard deviation, it was observed that the expectation is better approached with higher values of N (number of samples) than by using lower values of N and then repeating the simulation a high number of times, although the higher the value of N the longer the computation time and the greater the computer resources utilized. The standard deviation seemed to vary randomly and nothing significant could be observed from it.

	The Number of times for which the simulation is repeated					
N	1	500	1000	5000	10000	100000
600	3.1900	3.2200 Std. = 2.2671e-014	3.2600 Std. = 7.5977e-014	3.1900 Std. = 1.5722e-013	3.0600 Std. = 5.1917e-013	3.1400 Std. = 7.9146e-012
6000	3.1610	3.1090 Std. = 1.6448e-014	3.1260 Std. = 5.6872e-014	3.1800 Std. = 1.9720e-013	3.1590 Std. = 2.9356e-013	3.1610 Std. = 7.4163e-013
60000	3.1428	3.1394 Std. = 1.8670e-014	3.1456 Std. = 4.8430e-014	3.1501 Std. = 2.9890e-013	3.1379 Std. = 6.3197e-013	3.1483 Std. = 5.1746e-012

4.0 CONCLUSION

Monte Carlo methods are flexible and can take many sources of uncertainty, but they may not always be appropriate. In general, the method is preferable only if there are several sources of uncertainty.

5.0 REFERENCES

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